Math 564: Adv. Analysis 1 HOMEWORK 6 **Due: Dec 5 (Tue), 11:59pm**

1. Prove that for every monotone function $f : \mathbb{R} \to \mathbb{R}$, its set C_f of continuity points is cocountable (i.e. f is continuous at every point $x \in \mathbb{R} \setminus Q$ for some countable Q) and f' exists a.e.

HINT: It is enough to prove this for (not necessarily strictly) increasing functions.

2. Let $f : \mathbb{R} \to \mathbb{R}$. Prove:

(a) If f has bounded variation, then $T_f + f$ and $T_f - f$ are increasing.

HINT: To show that a function g is increasing, you need to show that $g(b) - g(a) \geqslant 0$ for all $a < b$.

(b) Conclude that f has bounded variation if and only if $f = g_+ - g_-$ for some bounded increasing functions g_+ and g_-.

(c) Deduce that f is a distribution of a (unique) finite Borel signed measure ν if and only if f is right-continuous and has bounded variation. (Recall that we proved $\Rightarrow$ in class, so you only need to prove $\Leftarrow$.)

3. Prove that for a function $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent:

(1) f is a distribution of a (unique) finite Borel signed measure $\nu \ll \lambda$.

(2) f' exists a.e. and is in $L^1(\lambda)$, and the fundamental theorem of calculus holds: for each $a < b$,

$$f(b) - f(a) = \int_a^b f' \, d\lambda.$$

(3) f has bounded variation and is absolutely continuous.

INSTRUCTIONS: You may use any results/proofs from class, as well as Folland's proofs (it is all there, if you can decode). You may also be concise in your proofs, just sketch them.

4. Let X, Y be normed vector spaces. Prove that if Y is a Banach space then so is $L(X, Y)$, the space of bounded linear transformations $X \to Y$.

5. [*Optional*] Let (X, μ) be a measure space and put $L^\infty := L^\infty(X, \mu)$. Prove:

(a) $\|\cdot\|_\infty$ is a norm on L^∞.

(b) L^∞ is a Banach space.

(c) Simple functions are dense in L^∞.

(d) $\lim_{q \to \infty} \|f\|_q = \|f\|_\infty$ for each $1 \leqslant p < \infty$ and $f \in L^p \cap L^\infty$.

HINT: Show that $\alpha_q \|f\|_\infty \leqslant \|f\|_q \leqslant \beta_q \|f\|_\infty$ for some $\alpha_q, \beta_q \to 1$ as $q \to \infty$. For the lower bound, use Chebyshev's inequality.

6. [*Optional*] Let (X,μ) be a measure space. Let $0 < p, q < \infty$ and $f \in L^p(X,\mu)$, $g \in L^q(X,\mu)$. Prove that the equality $\|fg\|_r = \|f\|_p \|f\|_q$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, holds in Hölder's inequality if and only if one of $|f|^p$ and $|g|^q$ is a scalar multiple of the other. What happens if p or q is infinite?

7. [*Optional*] Let (X,μ) be a measure space such that there is $m > 0$ with the property that every positive measure set has measure at least m. (E.g. counting measure.) Prove that for each $0 < p \leqslant q \leqslant \infty$, we have $L^p(X,\mu) \subseteq L^q(X,\mu)$, in fact,

$$\|f\|_q \leqslant m^{-(\frac{1}{p}-\frac{1}{q})} \, \|f\|_p.$$

When does the equality hold?